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# Entropy, time-irreversibility and the Schrödinger equation in a primarily discrete spacetime

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## Abstract

In this paper we show that the existence of a primarily discrete spacetime may be a fruitful assumption from which we may develop a new approach of statistical thermodynamics in pre-relativistic conditions. The discreteness of spacetime structure is determined by a condition that mimics the Heisenberg uncertainty relations, and the motion in this spacetime model is chosen as simple as possible. From these two assumptions we define a path entropy that measures the number of closed paths associated with a given energy of the system preparation. This entropy has a dynamical character and depends on the time interval on which we count the paths. We show that there exists a like-equilibrium condition for which the path entropy corresponds exactly to the usual thermodynamic entropy and, more generally, the usual statistical thermodynamics is reobtained. This result derived without using the Gibbs-ensemble method shows that the standard thermodynamics is consistent with a motion that is time irreversible at a microscopic level. From this change of paradigm it becomes easy to derive an *H-theorem*. A comparison with the traditional Boltzmann approach is presented. We also show how our approach can be implemented in order to describe reversible processes. By considering a process defined simultaneously by initial and final conditions, a well-defined stochastic process is introduced and we are able to derive a Schrödinger equation, an example of time-reversible equation.

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## 1. Introduction

Despite the large success obtained in statistical physics during the last century, we have not been able to derive the second law of thermodynamics from statistical mechanics although a large number of works have been devoted to this problem (for a review see, for

instance, [1]). In the Clausius version, the second law asserts the existence of a state function, the entropy, that is a non-decreasing function of time for any closed system [2]. The difficulty in the derivation of this law is the inadequacy between the postulated time-reversible behaviour for the motion of particles at a microscopic level and the observable-irreversible behaviour of macroscopic systems. In classical mechanics, we have to deal with fundamental questions as the existence of the Poincaré recurrence time [1]. In quantum physics, Landau and Lifshitz [3] suggested that the origin of irreversibility might be related to the measurement process that introduces a difference between past and future. Another kind of approach consists of associating the thermodynamic arrow of time with the cosmological expansion of the universe (see, for instance, [4]).

Since the problem of irreversibility resists to any demonstration by more than one century, it is tempting to investigate this problem starting from a new point of view. In a discussion about the foundations of statistical thermodynamics in terms of path integrals, Feynman [5] developed a given number of fundamental remarks leading to the question whether the existence of the Hamiltonian is needed to formulate classical statistical mechanics, i.e. the classical limit of quantum statistical physics. In parallel, a possible new foundation of statistical mechanics directly in terms of path integral was conjectured. The elaborations of statistical physics without Hamiltonian represent a new field of investigation [6]. If we adopt this point of view we have no longer to fight with the Poincaré recurrence time [1] in classical physics, but in quantum physics the Schrödinger equation cannot be used as a starting point and the canonical form of the density matrix must be abandoned. Consequently new physical ingredients have to be introduced, but where do we find them?

We know that the description of the universe at the Planck scale requires a deep modification of our usual physical concepts. For instance, in this domain a promising attempt consists in replacing point particles by strings or using a discrete spacetime structure instead of the differentiable manifold of the general relativity . . . [7, 8]. It is very tempting to see if such new postulates may have a translation, at least in a simplified version, in a pre-relativistic conditions and may lead to a new approach of statistical thermodynamics at a scale much larger than the Planck one. In addition, we may note that the combination of gravitation and quantum physics has led to the development of new thermodynamic concepts as the holographic principle [9] or the existence of a geometric entropy connected with the properties of the quantum vacuum. Such an entropy represents a very important discovery that leads to the description of fundamental thermodynamic laws from vacuum fluctuations without any reference to a Gibbs-ensemble description [10]. Thus, it is interesting to see if the concept of geometric entropy may be pertinent in a pre-relativistic world.

This paper represents an attempt in which we want to show that the existence of a primarily discrete spacetime may be a useful point of view from which we may develop a new approach of statistical thermodynamics.

Our approach is built up in several steps:

- (1) We assume the existence of a discrete spacetime for which the structure is determined by relations that mimic the Heisenberg uncertainty relations. Fortunately, to describe the usual thermodynamics the continuous limit of this spacetime model is sufficient.
- (2) We assume that the motion of a particle in spacetime is as simple as possible. This motion can be characterized by a real-valued function describing the transition from one spacetime point to another.
- (3) Amongst all the quantities that we can introduce to characterize the spacetime structure and the associated dynamics, we choose one that is similar to the standard thermal entropy. This path entropy becomes identical to the thermal one if the energy needed in the system preparation equilibrates the mean energy calculated on the paths. This result

obtained without any reference to the Gibbs-ensemble method shows that the standard thermodynamics is consistent with a motion at a microscopic level that is time irreversible.

- (4) From the previous points it is easy to derive an H-theorem without any new assumption in the case of free particles. At first glance our derivation of an H-theorem is very different from the Boltzmann one; nevertheless some comparisons are possible.
- (5) To be really convincing we must also be able to describe some situations that we may consider as reversible. Our approach is then implemented by the introduction of entry–exit conditions that create a past/future symmetry. Then we may describe the system by a complex-valued function that verifies a Schrödinger equation, i.e. a time-reversible equation in the Wigner sense.

Note that only points 1 and 2 correspond to real hypotheses.

The paper is organized as follows. In section 2 we introduce and discuss a spacetime model. In section 3 we characterize the spacetime by a path entropy and a path temperature that mimic the corresponding thermodynamic quantities. In section 4 we introduce a like-equilibrium condition from which the previous entropy and temperature coincide exactly with the corresponding thermodynamic quantities. In section 5 an H-theorem is demonstrated in the case of free particles and a comparison with Boltzmann derivation is presented. In the next section the approach is implemented in order to derive the Schrödinger equation. In the last section some comments and concluding remarks are presented.

## 2. Model of a primarily discrete spacetime

### 2.1. Introduction of the model

The choice between a discrete and a continuous version of the spacetime structure has already been analysed by Riemann in the classical world and more than 30 years ago Feynman presented some doubts concerning the continuum nature of spacetime in the quantum domain [11]. Today it is well accepted that the conventional notions of space and time break down at the Planck scale. Due to this various attempts to elaborate a quantum theory based on the existence of a discrete spacetime have been proposed (see, for instance, [8, 12] and [13] and the references quoted therein).

A discrete spacetime means that any length is built up from a finite number of the elementary length,  $\Delta x$ , and any time interval results from a series of individual ‘ticks’ of duration  $\Delta t$ . If a relation between  $\Delta x$  and  $\Delta t$  is expected, its precise form must depend on the accuracy with which we want to describe the world. To establish general aspects of the spacetime structure it is possible to start from new uncertainty relations issued from string theory (see, for instance, [14]) or simple *gedanken experiments* (see [15] for a short review in this field). These relations show a minimum of uncertainty in the determination of positions that is interpreted as the existence of a minimum  $\Delta x$  in the distance between two points while  $\Delta t$  is the minimum time interval needed to characterize two separate points. In the pre-relativistic domain that we consider the velocity of light is assumed to be infinite and there is no gravitational effect and hence the only one universal constant that we have to consider at the microscopic level is  $\hbar$ . Thus, if a mass  $m$  is located in a region  $\Delta x$  the only one relation that we can introduce between  $\Delta x$  and  $\Delta t$  is  $(\Delta x)^2/\Delta t = \hbar/m$ . Whatever the values of  $\Delta x$  and  $\Delta t$ , we have immediately  $\Delta x \Delta p = \hbar$  and  $\Delta t \Delta E = \hbar/2$  provided we use  $\Delta p = m \frac{\Delta x}{\Delta t}$  and  $\Delta E = (1/2)m \left(\frac{\Delta x}{\Delta t}\right)^2$ . Thus our assumption  $(\Delta x)^2/\Delta t = \hbar/m$  is equivalent to supposing that the fine structure of spacetime is described by relations that mimic the Heisenberg uncertainty relations. This seems a natural choice; it is reasonable to expect that the discrete spacetime

structure contains some ingredients that are already familiar to us from our study of the world at a larger scale [16]. This leads to a new kind of correspondence principle between a discrete spacetime and its continuum limit [17].

However, the relation  $(\Delta x)^2/\Delta t = \hbar/m$  or the uncertainty relations do not fix the precise values of  $\Delta x$  and  $\Delta t$ . If we want to avoid particle creation by quantum fluctuations,  $\Delta x$  must be larger than the Compton wavelength ( $\hbar/mc$ ) and  $\Delta t \gg \hbar/mc^2$ , where  $c$  is the velocity of light; thus, in a pre-relativistic world ( $c \rightarrow \infty$ ) there is no limitation from below for  $\Delta x$  and  $\Delta t$ . Hence we are free to assume that both  $\Delta x$  and  $\Delta t$  tend to zero provided we keep the relation  $(\Delta x)^2/\Delta t = \hbar/m$  in this limit. This is the starting point of our approach.

## 2.2. Dynamics in spacetime

Although this is probably not needed, in this first attempt, we assume that the spacetime points  $(t_i, x_i)$  are located on the sites of a regular lattice, as in the chessboard problem investigated in [5]. Here we assume that the motion is as simple as possible. A particle may jump, at random, from one site to one of its nearest neighbours. Thus, by definition, a path corresponds to a set of sites  $(t_i, x_i)$ ; the values of  $t_i$  are such as  $t_{i+1} > t_i$  whatever be the value of  $i$  and the coordinate positions, and  $x_{i+1}$  is necessarily one of the nearest neighbours of  $x_i$ . The conditions over  $\Delta x$  and  $\Delta t$  defined above lead, in the limit  $\Delta x, \Delta t \rightarrow 0$ , to a continuous diffusion process [18] for which the diffusion coefficient is  $D = \hbar/2m$ . This diffusion process has a pure quantum origin,  $D \rightarrow 0$  if  $\hbar \rightarrow 0$ . The random walk can be characterized by a real-valued continuous function,  $q_0(t_0, x_0; t, x)$ , representing the density of transition probability to go from  $(t_0, x_0)$  to  $(t, x)$  when  $t \geq t_0$ . From  $q_0(t_0, x_0; t, x)$  and a function  $\phi_0(x)$  defined for  $t = t_0$  we form the function  $\phi(t, x)$  according to

$$\phi(t, x) = \int \phi_0(y)q_0(t_0, y; t, x) dy \quad (1)$$

which is the solution of the diffusion equation

$$-\partial\phi(t, x)/\partial t + D\Delta_x\phi(t, x) = 0 \quad (2)$$

verifying the initial-value problem  $\phi(t_0, x) = \phi_0(x)$ . Note that  $q_0(t_0, x_0; t, x)$  is the fundamental solution of (2) in which  $\Delta_x$  is the Laplacian operator taken at the point  $x$ . In the presence of an external potential,  $u(t, x)$ , we generalize (2) to

$$-\partial\phi(t, x)/\partial t + D\Delta\phi(t, x) - \frac{1}{\hbar}u(t, x)\phi(t, x) = 0. \quad (3)$$

In contrast with (2), the fundamental solution of (3),  $q(t_0, x_0; t, x)$ , cannot be normalized in general [19]. Thus,  $q(t_0, x_0; t, x)$  is no longer a transition probability density but it verifies the Chapman–Kolmogorov law of composition [19], and therefore it can be used to describe transitions in spacetime. By using the Feynman–Kac formula, the fundamental solution of (3) can be written in terms of path integral. Then  $q(t_0, x_0; t, x)$  appears as a weighted sum of all the paths  $x(t)$  connecting the spacetime points  $(x_0, t_0)$  to  $(x, t)$ ; the weight of a path is determined by

$$A[x(t); t, t_0] = \int_{t_0}^t \left[ \frac{1}{2}m \left[ \frac{dx(t')}{dt'} \right]^2 + u(t', x(t')) \right] dt', \quad (4)$$

and we have

$$q(t_0, x_0; t, x) = \int \mathcal{D}x(t) \exp -\frac{1}{\hbar}A[x(t); t, t_0] \quad (5)$$

where  $\mathcal{D}x(t)$  means the measure for the functional integral. Note that the integrand in (4) looks like the Hamiltonian for a free particle in the presence of an external potential and therefore  $A[x(t); t, t_0]$  will be called the Hamiltonian action. However, the continuous notation for  $A[x(t); t, t_0]$  must be considered as formal [20]; to calculate the path integral we have to discretize  $A[x(t); t, t_0]$  and the paths that contribute to the integral are those for which there is no derivative, i.e. no velocity in the usual sense [21].

### 2.3. Relevance of the primarily discrete spacetime concept

In the continuous limit our primarily discrete spacetime leads to (3), therefore it should be possible to start our derivation of statistical thermodynamics directly from this diffusion equation. But in that case we have to explain why to introduce a diffusion process. Here we do not give an answer to this question but we shift it to a more fundamental one, i.e. what is the origin of the uncertainty relations. Starting with this point of view we encounter a problem existing in other domains of quantum physics, and due to this it becomes possible to translate some concepts from one domain to another. Because we have elaborated a formalism reminiscent of the one introduced at the Planck scale, it is normal to consider the possibility of recovering at the level of standard thermodynamics some results obtained at the Planck scale. This is what we will do below and, in addition, we shall see that this approach allows us to derive an *H-theorem*.

Note also that our model is well defined in the following sense. It is well known that the trajectories associated with a diffusion process are fractal and then if we start directly from a diffusion equation we have to deal with the following dilemma [22]: is the geometry of the underlying spacetime fractal or is the underlying spacetime regular and the fractal character generated by the dynamics? Here a clear choice has been made and for the properties investigated we do not need to introduce a fractal spacetime as suggested by some authors [22]. Our spacetime has the same meaning as the one introduced by Feynman in his seminar paper [23] on non-relativistic systems but in addition we have the uncertainty relations, introduced at the level of the lattice. As we shall see, since we are able to derive the Schrödinger equation and then the usual Heisenberg uncertainty relations, we have an illustration here of the correspondence principle invoked in the causal set theory [17].

Thus, the introduction of a primarily discrete spacetime appears as a unifying concept.

## 3. Path entropy

*A priori*, to describe structure and dynamics in our spacetime model we may introduce a lot of functions. However, since our main goal is to recover thermodynamics we focus on quantities that resemble as far as possible the thermodynamic ones. To describe equilibrium situations we restrict the external potential,  $u(t, x)$ , to be a time-independent quantity,  $u(x)$ . Accordingly,  $A[x(t); t, t_0]$  becomes  $A[x(t); \tau]$  with  $\tau = t - t_0$ . By analogy with the entropy that is the key quantity in thermodynamics we have to introduce a quantity that measures the order in spacetime. Since the standard entropy is defined for given values of internal energy and volume we must consider that our spacetime system is prepared with a given energy  $U$  and occupies a volume  $V$ .

To characterize the spacetime order around each point  $x_0$  we count the number of paths for which  $x(\tau) = x_0$  and on which the Hamiltonian action,  $A[x(t); \tau]$ , does not deviate too much

from the action  $\tau U$ . This is realized by introducing the weight  $\exp -\frac{1}{\hbar}[A[x(t); \tau] - \tau U]$ . By analogy with the standard thermodynamics we define a path entropy,  $S_{\text{path}}$ , according to

$$S_{\text{path}} = k_B \ln \int dx_0 \int \mathcal{D}x(t) \exp -\frac{1}{\hbar}[A[x(t); \tau] - \tau U]. \quad (6)$$

The larger the  $S_{\text{path}}$ , the larger the number of paths, which means the larger the acceptable fluctuations of the Hamiltonian action around  $\tau U$  and less strict is the spacetime order. Thus  $S_{\text{path}}$  can be used to characterize the order in spacetime. However,  $S_{\text{path}}$  depends on  $\tau$ , that is, for the moment, a free parameter, and due to this we cannot conclude that  $S_{\text{path}}$  corresponds to the thermodynamic entropy.  $S_{\text{path}}$  can also be rewritten as

$$S_{\text{path}} = \frac{k_B \tau}{\hbar} U + k_B \ln Z_{\text{path}} \quad (7)$$

with

$$Z_{\text{path}} = \int dx_0 \int \mathcal{D}x(t) \exp -\frac{1}{\hbar} A[x(t); \tau] = \int dx_0 q(0, x_0; \tau, x_0) \quad (8)$$

in which we have taken  $t_0 = 0$  and  $q(0, x_0; \tau, x_0)$  is the fundamental solution of (3) for closed paths and a time interval  $\tau$ .  $Z_{\text{path}}$  is the total number of closed paths that we may count during  $\tau$  irrespective of the value of  $U$ . Now we may choose  $\tau$  as a function of  $U$ .

With  $S_{\text{path}}$  we may associate a path temperature  $T_{\text{path}}$  by  $\frac{dS_{\text{path}}}{dU} = \frac{1}{T_{\text{path}}}$ , which shows the dependence of  $S_{\text{path}}$  on  $U$ . Similar to  $S_{\text{path}}$ ,  $T_{\text{path}}$  is a well-defined quantity but it cannot be identified with the thermodynamic temperature at this level. From (7) we immediately have

$$\frac{1}{T_{\text{path}}} = \frac{k_B \tau}{\hbar} + \frac{k_B}{\hbar} \left[ U + \hbar \frac{d}{d\tau} \ln Z_{\text{path}} \right] \frac{d\tau}{dU}. \quad (9)$$

To calculate  $\frac{d}{d\tau} \ln Z_{\text{path}}$  we use the expression of  $Z_{\text{path}}$  in terms of  $q(0, x_0; \tau, x_0)$  given in (8) and from the Chapman–Kolmogorov law of composition we may write

$$q(0, x_0; \tau, x_0) = \int dx_b q(0, x_0; \delta t, x_b) q(\delta t, x_b; \tau - \delta t, x_0) \quad (10)$$

for any  $\delta t$  such as  $0 < \delta t < \tau$ . From (8), (10) and (3) we have

$$\hbar \frac{d}{d\tau} \ln Z_{\text{path}} = \frac{\hbar}{Z_{\text{path}}} \int dx_0 \int dx_b q(0, x_0; \delta t, x_b) \left[ D \Delta_{x_0} - \frac{1}{\hbar} u(x_0) \right] q(\delta t, x_b; \tau - \delta t, x_0). \quad (11)$$

This equation shows that two contributions are expected in the calculation of  $\hbar \frac{d}{d\tau} \ln Z_{\text{path}}$ . One is associated with the external potential,  $u(x_0)$ , which can be written as

$$\frac{1}{Z_{\text{path}}} \int dx_0 u(x_0) q(0, x_0; \tau, x_0) = \langle u_P \rangle_{\text{path}}. \quad (12)$$

We may interpret  $\langle u_P \rangle_{\text{path}}$  as the potential energy calculated as an average over the paths. The second contribution to  $\hbar \frac{d}{d\tau} \ln Z_{\text{path}}$  is related to the Laplacian operator that enters in (11). To calculate this term we use the fact that (10) holds if  $\delta t \rightarrow 0$ ; in this limit we may neglect the external potential in the Hamiltonian action and  $\Delta_{x_0}$  operates only on the term which looks like the kinetic energy in  $A[x(t); \tau]$ . A simple calculation gives

$$\frac{1}{Z_{\text{path}}} \int dx_0 \int d(\delta x) q(0, x_0; \delta t, x_0 + \delta x) \left[ \frac{m}{2} \left( \frac{\delta x}{\delta t} \right)^2 - \frac{\hbar}{2\delta t} \right] q(\delta t, x_0 + \delta x; \tau - \delta t, x_0), \quad (13)$$

which we can rewrite as

$$\frac{m}{2} \left\langle \left( \frac{\delta x}{\delta t} \right)^2 \right\rangle_{\text{path}} - \frac{\hbar}{2\delta t}. \quad (14)$$

This shows that the Laplacian operator in (11) leads to the two terms shown in (14); they represent the difference between the kinetic energy calculated as an average over the paths and the quantum fluctuation of energy corresponding to a time interval  $\delta t$  on which the kinetic energy is calculated. It is easy to prove that (14) is finite in the limit  $\delta t \rightarrow 0$ . In this limit it is legal to replace  $q(0, x_0; \delta t, x_0 + \delta x)$  in (13) by the free-particle approximation that only depends on  $\delta t$  and  $\delta x$ . Moreover, we may approximate  $q(\delta t, x_0 + \delta x; \tau - \delta t, x_0)$  by  $q(0, x_0; \tau, x_0)$  and its integral over  $x_0$  gives  $Z_{\text{path}}$  as shown in (8). In the limit that we consider it is easy to see that  $\frac{m}{2} \left\langle \left( \frac{\delta x}{\delta t} \right)^2 \right\rangle_{\text{path}} = \frac{\hbar}{2\delta t}$  (see also [21]). This result is also expected from the properties of the initial lattice on which we have  $\frac{1}{2} m \left( \frac{\delta x}{\delta t} \right)^2 = \frac{\hbar}{2\delta t}$ . Thus the quantity (14) is well defined in the limit  $\delta t \rightarrow 0$ . More generally, we write

$$\frac{m}{2} \left\langle \left( \frac{\delta x}{\delta t} \right)^2 \right\rangle_{\text{path}} = \frac{\hbar}{2\delta t} - \langle u_K \rangle_{\text{path}} \quad (15)$$

in which  $\langle u_K \rangle_{\text{path}}$  is a well-behaved function in the limit  $\delta t \rightarrow 0$ . In principle  $\langle u_K \rangle_{\text{path}}$  and  $\langle u_P \rangle_{\text{path}}$  can be calculated for a given potential  $u(x)$  and a given value of  $\tau$ . Indeed, we have checked the previous results by performing an exact calculation but using an explicit form of  $u(x)$ . Finally, we can rewrite (9) according to

$$\frac{\hbar}{k_B T_{\text{path}}} = \tau + [U - (\langle u_K \rangle_{\text{path}} + \langle u_P \rangle_{\text{path}})] \frac{d\tau}{dU}. \quad (16)$$

This equation establishes a relation between  $\tau$  and  $T_{\text{path}}$  when  $u(x)$  and  $U$  are given.

#### 4. Thermodynamics

The square bracket in (16) contains  $U$ , the specific properties of the system via the averages over the paths and the variable  $\tau$ . We are free to choose a particular value,  $\tau^*$ , of  $\tau$  in such a way that the quantity

$$[\langle u_K \rangle_{\text{path}} + \langle u_P \rangle_{\text{path}}], \quad (17)$$

which looks like the internal energy calculated over the paths, is indeed equal to the energy of the system preparation, i.e.  $U$ ; this is clearly an equilibrium condition since the energy that is brought to the system,  $U$ , is transformed into the sum of the mean kinetic and potential energy associated with the paths. Thus, whatever the value of the derivative  $\frac{d\tau^*}{dU}$  from (16), we get  $\tau^* = \frac{\hbar}{k_B T^*} = \beta^* \hbar$ , which fixes the value of  $T^*$  when  $\tau^*$  is given. The partition function (8) can be rewritten as

$$Z_{\text{path}}^* = \int dx_a \int \mathcal{D}x(t) \exp -\frac{1}{\hbar} \int_0^{\beta^* \hbar} \left[ \frac{1}{2} m \left[ \frac{dx(t')}{dt'} \right]^2 + u(x(t')) \right] dt'. \quad (18)$$

This expression is identical to that of the standard partition function  $Z$  obtained in [5] starting from the canonical form of the density matrix provided  $T^*$  is identified with the thermal temperature  $T$ . This is justified because our definitions of  $S_{\text{path}}$  and  $T^*$  are in agreement with the zeroth law of thermodynamics. In order to prove this point we follow the Callen derivation of the zeroth law [24]. Let us consider a system formed by two independent subsystems  $S_1$  and  $S_2$ ; they may differ by both the mass and the external potential and their preparation energy



corresponds to  $U_1$  and  $U_2$ . The equilibrium condition defined above can be written for the global system and for each subsystem taken separately. With our definition of  $S_{\text{path}}$  it is easy to see that the entropy is the sum of entropies of each subsystem. Now for a fixed value of  $U = U_1 + U_2$  we may imagine a virtual energy transfer  $\delta U$  from one system to the other. The requirement that, at the equilibrium, the total entropy is maximum relative to any  $\delta U$  leads to the conclusion that the two subsystems must have the same value for  $T^*$ , in agreement with the zeroth law. The equilibrium condition also leads to the relation  $\frac{d}{d\beta} \ln Z_{\text{path}}^* = -U$ , which is expected from standard statistical mechanics [2]. If we define  $F$  according to  $F = -k_B T \ln Z$  we see that (7) is nothing else than the standard relation  $F = U - T S_{\text{path}}^*$  and consequently  $S_{\text{path}}^*$  corresponds to the traditional thermal entropy.

Thus we have obtained an equivalent description of the thermodynamics directly in terms of paths without using the Gibbs-ensemble method. In this dynamic approach the motion associated with paths is not due to a Hamiltonian but results from the primarily discreteness of the spacetime. However, our approach is more than a simple alternative description of thermodynamics as we shall see below. (For brevity in what follows we drop the superscript \*.)

In standard path integral formalism  $\beta\hbar$  is considered as a *formal time* [5]. Here,  $\tau$  results from a combination of dynamics determined by (3) and thermodynamics via a particular choice in the solution of (16), therefore  $\tau$  must have a strong physical meaning. From standard textbooks in statistical mechanics [2], it is well known that there is no entropy on a short period of time. To have thermodynamics we must consider time intervals such that the quantum fluctuations do not exceed the order of magnitude of the typical thermal energy. In the case of free particles the mean value of the kinetic energy is  $1/2\beta$  and from the time-energy uncertainty relation ( $\tau/2\beta = \hbar/2$ ) we have  $\tau = \beta\hbar$ ; we see that  $\tau$  represents the relaxation time that we have to take in order to relax the quantum fluctuations and to reach the thermal regime. Our derivation of  $\tau$  is more general since it is independent of the external potential, but it has the same physical meaning. In parallel, using the results given in [21], it is easy to see that the paths are located, on average, on a given volume in space; in the absence of external potential this volume is a sphere of radius  $\Lambda = \frac{\hbar}{(mk_B T)^{1/2}}$  corresponding to the thermal de Broglie wavelength, an expected result. However, the most important result of our approach is that equilibrium thermodynamics is consistent with a motion that is time irreversible at the microscopic level as given by (3). The interest of this result is to open a door from which we may establish an *H-theorem* without any new assumption.

## 5. Derivation of an *H-theorem*

In the Boltzmann approach of the *H-theorem* [1] a system of particles is first prepared in a non-equilibrium state by external constraints. At the time  $t = t_0$  these constraints are removed and the system relaxes towards its equilibrium state. The Clausius version of the second law of thermodynamics asserts that during the system relaxation there exists a function that increases monotonically versus  $(t - t_0)$  and tends to the thermal entropy when  $(t - t_0)$  becomes infinite. In the Boltzmann work there is no external potential for  $t \geq t_0$ ; the relaxation is driven by collisions between particles. Hereafter we mimic a similar description in our spacetime model.

Using an external potential we prepare the system in such a way that, at the initial time,  $t = t_0$ ; the space points are weighted by a distribution function,  $\phi_0(x)$ , which is positive and normalized. Any quantity calculated in these conditions depends on  $\phi_0(x)$ , at least for  $(t - t_0) \geq 0$  but finite. For  $t \geq t_0$  the external potential is switched off,  $u(x) = 0$ , and the time evolution of  $\phi(t, x)$  is given by (2). In this case  $q(t_0, x_0; t, x)$  is a density of transition

probability for  $t > t_0$  and  $\phi(t, x)$  defined according to (1) is the probability of being in  $x$  at time  $t$ .

Now, we introduce a function  $H(t)$  that generalizes  $S_{\text{path}}$  given by (6). After relaxation, the space is uniform since  $u(x) = 0$  and the entropy must be given by (6) that we can rewrite as  $S = k_B \ln(V\gamma(\tau))$ , where  $V$  is the volume of the system. During the relaxation process, at each time  $t$ , and for each point  $x$ , we count the number of closed paths that we can form between the instants  $t$  and  $t + \tau$ ; this number is still  $\gamma(\tau)$ . However, the total number of paths at this time and this position is  $\gamma(\tau)/\phi(t, x)$ ; it reduces to  $\gamma(\tau)V$  if  $\phi(t, x) = 1/V$ , i.e. in the case of a uniform system. We define the average,  $S_{\text{total}}(t)$ , of  $\ln[\gamma(\tau)/\phi(t, x)]$  taken over the overall volume as

$$S_{\text{total}}(t) = k_B \int \phi(t, x) \ln \left[ \frac{\gamma(\tau)}{\phi(t, x)} \right] dx. \quad (19)$$

Now we consider the quantity  $H(t) = (1/k_B)(S_{\text{total}}(t) - S)$  defined by

$$H(t) = - \int \phi(t, x) \ln[\phi(t, x)] dx - \ln[V] \quad (20)$$

where we have used the normalization of  $\phi(t, x)$ . The first term in (20) is positive since the integrand is negative due to the fact that  $0 \leq \phi(t, x) \leq 1$ . Thus,  $H(t)$  represents the competition between two quantities of opposite sign. The solution of (3) in a volume  $V$  has been given in [25]; in the limit  $(t - t_0) \rightarrow \infty$  it has been shown that  $\phi(t, x) = 1/V$  and consequently  $H(t) \rightarrow 0$ . In addition, we can show that [25]

$$\frac{dH(t)}{dt} = \left( \frac{\hbar}{2m} \right) \int \frac{1}{\phi(t, x)} \left( \frac{d\phi(t, x)}{dx} \right)^2 dx \geq 0. \quad (21)$$

Thus,  $H(t)$  is a *monotonic increasing* function of  $t$  and  $H(t)$  vanishes in the stationary regime obtained in the limit  $(t - t_0) \rightarrow \infty$ . the time is running,  $S_{\text{total}}(t)$  increases in a monotonic manner to reach a stationary value corresponding to the thermal entropy  $S$ . Thus an *H-theorem* is demonstrated.

The irreversible behaviour results from the existence of a spacetime, which is discrete at a fundamental level and induces a special dynamics not connected with the Hamiltonian. Nowhere do we consider collisions between particles as in the Boltzmann dynamics, but we investigate the paths performed by one mass. To characterize the dynamics on these paths we select a given point  $(x_i, t_i)$  and for a time interval  $\delta t \leq \tau/2$  we consider the forward and backward velocities defined respectively as  $V_+ = [(x_{i+1} - x_i)/\delta t]$  and  $V_- = [(x_i - x_{i-1})/\delta t]$ . With the product  $V_+V_-$  we associate [21] an average over paths  $\langle V_+V_- \rangle$ ; it has been shown that [21]

$$\langle V_+V_- \rangle = \left\langle \frac{(x_{i+1} - x_i)}{\delta t} \frac{(x_i - x_{i-1})}{\delta t} \right\rangle = -1/(m\beta). \quad (22)$$

This result, independent of  $\delta t$ , leads to several remarks. First, for trajectories on which there is a velocity in the usual sense  $\langle V_+V_- \rangle$  is a positive quantity; here the negative value of  $\langle V_+V_- \rangle$  is the signature of the fractal character of the paths. Second, with the negative value of  $\langle V_+V_- \rangle$  we may associate a *formal collision*, which reverses the direction of the velocity, on average. From (22) we see that the correlations are not destroyed after a formal collision in contrast with the molecular chaos hypothesis used by Boltzmann [1]. It is only in the limit  $T = 0$  that the velocities  $V_+$  and  $V_-$  become statistically independent. Third, a formal collision appears at any time and everywhere in space; in contrast to the Boltzmann dynamics we cannot consider two separate time scales, one for the duration of collisions and the other for the time between collisions.

Of course to be very satisfying our approach must also describe some situations that we may consider as reversible.

## 6. Time-reversible processes and the Schrödinger equation

In the previous sections we have considered the real-valued function

$$\phi(t, x) = \int \phi_0(y)q(t_0, y; t, x) dy \quad (23)$$

that is the solution of the diffusion equation (3) verifying the initial-value problem  $\phi(t_0, x) = \phi_0(x)$ ; in (23),  $q(t_0, x_0; t, x)$  is the fundamental solution of (3). This description is quite natural; we know the initial condition for  $t = t_0$  and describe the system for  $t \geq t_0$ . However, it is possible to consider another function  $\hat{\phi}(t, x)$  defined according to

$$\hat{\phi}(t, x) = \int q(t, x; t_1, y)\hat{\phi}_1(y) dy \quad (24)$$

for  $t_0 \leq t \leq t_1$ . Using the properties of the fundamental solution it is easy to prove that  $\hat{\phi}(t, x)$  is the solution of the differential equation

$$\frac{\partial \hat{\phi}(t, x)}{\partial t} + D\Delta \hat{\phi}(t, x) - \frac{1}{\hbar}u(t, x)\hat{\phi}(t, x) = 0 \quad (25)$$

verifying the final condition  $\phi(t_1, x) = \hat{\phi}_1(x)$ . Up to now the use of (3) or (25) is a question of convenience depending on the information we have on the system.

In what follows we consider a new physical situation in which the system is defined simultaneously by (3) and (25), i.e. by a dynamics defined at an initial and a final time,  $t_0$  and  $t_1$ , respectively. Of course, for *the entry–exit conditions* ( $\phi_0(x), \hat{\phi}_1(x)$ ) our problem is now to predict what happens for any  $t$  between  $t_0$  and  $t_1$ . This is reminiscent of the Lagrangian mechanics that is known to describe reversible processes. However, in our case we must show that the entry–exit conditions introduce a well-identified mathematical object as, for instance, a well-defined stochastic process. The proof of this has been given by Nagasawa [19], and we will not reproduce the mathematical details here but we will focus mainly on the physical aspects.

The functions  $\phi(t, x)$  and  $\hat{\phi}(t, x)$ , which describe the transition in spacetime, are not transition probability density and, up to now, there is no restriction on  $\phi_0(x)$  and  $\hat{\phi}_1(x)$ . Hereafter we assume that  $\phi_0(x)$  and  $\hat{\phi}_1(x)$  are two non-negative real-valued functions that we choose such as

$$\int \phi_0(y) dy q(t_0, y; t_1, x) dx \hat{\phi}_1(x) = 1. \quad (26)$$

From the set  $[\phi_0(x), \hat{\phi}_1(x), q(s, y; t, x)]$ , condition (26) and the Chapman–Kolmogorov law of composition, we can show that [19]

$$\mu(t, x) = \phi(x)\hat{\phi}(x) \quad (27)$$

is a non-negative and normalized quantity; we may consider  $\mu(t, x)$  as the probability distribution of a stochastic process. Moreover, from the same ingredients as above we can define a probability measure denoted as  $Q = [\phi_0 q] \langle \langle q \hat{\phi}_1 \rangle \rangle$  [19]. It is possible to put  $Q$  in a more traditional form by introducing the new variables

$$p(s, y; t, x) = \phi(s, y)q(s, y; t, x) \frac{1}{\phi(t, x)} \quad \text{if } \phi(t, x) \neq 0 \quad \text{and} \\ p(s, y; t, x) = 0 \quad \text{otherwise.} \quad (28)$$

and

$$\hat{p}(s, y; t, x) = \frac{1}{\hat{\phi}(s, y)}q(s, y; t, x)\hat{\phi}(t, x) \quad \text{if } \hat{\phi}(s, y) \neq 0 \quad \text{and} \\ \hat{p}(s, y; t, x) = 0 \quad \text{otherwise.} \quad (29)$$

From their definitions, we can see that  $p(s, y; t, x)$  and  $\hat{p}(s, y; t, x)$  are positive and normalized according to

$$\int p(s, y; t, x) dy = 1 \quad \text{and} \quad \int dx \hat{p}(s, y; t, x) = 1. \quad (30)$$

Consequently we can consider  $p(s, y; t, x)$  and  $\hat{p}(s, y; t, x)$  as transition probability densities. The probability measure  $Q$  can be rewritten as  $Q = [\phi_0 \hat{\phi}_0 \hat{p}]$ , which is the traditional Kolmogorov representation of the Markov process with the initial distribution  $\mu(t_0, x) = \phi_0(x) \hat{\phi}_0(x)$  and the transition probability density  $\hat{p}(s, y; t, x)$ .  $Q$  is also identical to the time-reversed Markov process  $Q = \langle \langle p \phi_1 \hat{\phi}_1 \rangle \rangle$ . Thus the probability measure denoted as  $Q = [\phi_0 q] \langle \langle q \hat{\phi}_1 \rangle \rangle$  is the Schrödinger representation of the Markov process. This representation has been introduced by Schrödinger [26] and later investigated by Kolmogorov and several other mathematicians (for a review in this field see [27]). In the Kolmogorov representation, the Markov process and its time-reversed version define two positive semigroups that are in duality relative to the measure  $\mu(t, x) dt dx$ . The transition probability densities  $p(s, y; t, x)$  and  $\hat{p}(s, y; t, x)$  verify diffusion equations

$$-\frac{\partial p(s, y; t, x)}{\partial t} + (\hbar/2m) \Delta_x p(s, y; t, x) + a(t, x) \nabla_x p(s, y; t, x) = 0 \quad (31)$$

and

$$\partial \hat{p}(s, y; t, x) \partial t + (\hbar/2m) \Delta_x \hat{p}(s, y; t, x) + \hat{a}(t, x) \nabla_x \hat{p}(s, y; t, x) = 0 \quad (32)$$

while the density distribution of the Markov process obeys the law

$$\frac{\partial \mu(t, x)}{\partial t} + \nabla \left( \frac{\hat{a} - a}{2} \mu(t, x) \right) = 0. \quad (33)$$

The drift functions  $a$  and  $\hat{a}$  are determined by the duality condition, which leads to the celebrated Kolmogorov result  $a(t, x) = (\hbar/m) \nabla_x \ln[\phi(t, x)]$  and  $\hat{a}(t, x) = (\hbar/m) \nabla_x \ln[\hat{\phi}(t, x)]$ .

Thus the process defined by the entry–exit conditions and equations (3) and (25) is the Schrödinger representation of a well-defined Markov process. This representation is useful to show the existence of a superposition principle for Markov processes.

Instead of the two real-valued functions  $\phi(t, x)$  and  $\hat{\phi}(t, x)$ , we can introduce two other real-valued functions

$$R(t, x) = \frac{1}{2} \ln \phi(t, x) \hat{\phi}(t, x) \quad \text{and} \quad S(t, x) = \frac{1}{2} \ln \frac{\hat{\phi}(t, x)}{\phi(t, x)} \quad (34)$$

which we can combined into one complex-valued function  $\Psi(t, x) = \exp[R(t, x) + iS(t, x)]$ . It has been shown [19] that  $\Psi(t, x)$  verifies the Schrödinger equation

$$i\hbar \frac{\partial \Psi(t, x)}{\partial t} = -\frac{\hbar^2}{2m} \Delta_x \Psi(t, x) + V(t, x) \Psi(t, x) \quad (35)$$

in which  $V(x, t)$  is related to  $u(t, x)$  according to

$$V(t, x) - u(t, x) + 2\hbar \left[ \frac{\partial S(t, x)}{\partial t} + D(\nabla S)^2(t, x) \right] = 0. \quad (36)$$

Thus, when we go from the representation involving two real-valued functions to the other one based on one complex-valued function  $u(t, x)$  has to be changed into  $V(t, x)$  as shown in (36). In addition to the previous results the equation for the complex conjugate  $\bar{\Psi}(t, x)$  of  $\Psi(t, x)$  has been established and we have the basic result

$$\mu(t, x) = \phi(t, x) \hat{\phi}(t, x) = \Psi(t, x) \bar{\Psi}(t, x) \quad (37)$$

which gives the physical interpretation of the product  $\Psi(t, x)\bar{\Psi}(t, x)$ . Finally, the superposition principle of Markov processes shows that  $\Psi(t, x)$  verifies the usual superposition principle associated with a wavefunction. Since the Schrödinger equation is time reversible in the Wigner sense, in this section we have shown how to implement the description presented in the previous sections in order to be able to describe reversible processes.

The derivation of the Schrödinger equation that we have developed here is totally different from a recent approach in which the complex nature of the wavefunction is connected with the assumption that the spacetime, by itself, has a fractal nature [28]. Another route to derive the Schrödinger equation has been developed in a series of very interesting papers presented by Ord (see, for instance, [29, 30]). In these papers the role of time irreversibility is carefully analysed and the reversibility of the Schrödinger equation is obtained by considering more information on the paths but selecting a special projection of the processes that appears as time reversible. Here the route that we have retained is the one that allows us to treat on a similar footing statistical mechanics and quantum physics. This route is also in the spirit of standard thermodynamics in which it is claimed that the physics is basically irreversible but that we may create some reversible processes by imposing a symmetry between initial and final states [31].

## 7. Concluding remarks

The existence of a discrete spacetime is one of the concepts used to describe the physics at the Planck scale. In this paper we show that the existence of a primarily discrete spacetime can also be a powerful concept to describe the physics in a pre-relativistic world. Our work is based on two main assumptions: (i) the discrete spacetime structure is determined by relations that mimic the Heisenberg uncertainty relations and (ii) the motion in this spacetime model is as simple as possible. For the properties that we have in mind the continuous limit is sufficient. All quantities defined in this approach have a dynamical character; they depend on the dynamics on the paths and on the time interval,  $\tau$ , on which we observe the paths. We have shown that there exists a characteristic time interval,  $\tau = \beta\hbar$ , for which the mean value of the energy calculated over the closed paths corresponds exactly to the energy needed in the system preparation; for this like-equilibrium condition our results are identical to the standard ones expressed in terms of path integral. We have seen that  $\tau$  represents the time that we have to take in order that the quantum fluctuations do not exceed the thermal ones. Of course, it is possible to follow the system evolution for time intervals smaller than  $\tau$ , but in this case we do not have the usual thermodynamics.

Our approach relates thermodynamics to the existence of motions that are not time reversible at the microscopic level. From this change of paradigm it becomes easy to derive an *H-theorem*. With the motion in spacetime we associate formal collisions; they may appear everywhere and at any time as a consequence of the fractal character of the trajectories. Moreover, in contrast with the Boltzmann dynamics, a formal collision does not destroy the correlations in the velocities.

At least in the simple example investigated here we may say that we have demonstrated the Feynman conjecture, i.e. the possibility of deriving the statistical thermodynamics directly from the inspections of the paths and without using all the apparatus of quantum mechanics. In this respect there is a parallel with the black-hole thermodynamics for which it has been shown that the main results can be derived from general features of the theory rather than the detailed form of the Einstein equation [32]. In addition, it is interesting to underline that our  $S_{\text{path}}$  is basically related to the spacetime properties; it does not result from the counting of microstates as in standard thermodynamics. However, we can perform an exact mapping between  $S_{\text{path}}$

and the thermal entropy; a similar mapping exists between the standard heat-bath formalism and the thermal behaviour induced by vacuum polarization in the presence of causal horizons [10]. Of course, here we have no quantum vacuum and no horizon and therefore  $S_{\text{path}}$  has not the same origin as that of the geometrical entropy introduced in the vicinity of a black hole; nevertheless from our approach we may emphasize some similarities between two different fields of investigations.

The results obtained in statistical thermodynamics are based on the properties of a real-valued function  $\phi(t, x)$  that verifies a diffusion-like equation (3), but this equation is not a simple imaginary-time version of the Schrödinger equation; no analytic continuation is invoked in our approach. To get the Schrödinger equation we force the system to have a time-reversible behaviour for any time; from this we leave statistical thermodynamics and enter quantum mechanics.

Thus the existence of a primarily discrete spacetime appears as a unifying concept from which we may relate some results existing in different domains and organize a simple transition between statistical thermodynamics and quantum physics.

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